7.3.1 Mixing Noise with Sinusoid

A situation often encountered in communication systems is one in which noise is mixed with (i.e. multiplied by) a deterministic sinusoidal waveform. Let the sinusoidal waveform be $\cos 2\pi f_0 t$. Then the product of this waveform with a spectral noise component, as given by Eq. (7.15), yields

$$n_k(t)\cos 2\pi f_0 t = \frac{a_k}{2}\cos 2\pi(k\,\Delta f + f_0)t + \frac{b_k}{2}\sin 2\pi(k\,\Delta f + f_0)t + \frac{a_k}{2}\cos 2\pi(k\,\Delta f - f_0)t + \frac{b_k}{2}\sin 2\pi(k\,\Delta f - f_0)t \tag{7.37}$$

Thus, the mixing gives rise to two noise spectral components, one at the sum frequency $f_0 + k\,\Delta f$ and one at the difference frequency $f_0 - k\,\Delta f$. In addition, the amplitudes of each of the two noise spectral components generated by mixing has been reduced by a factor of 2 with respect to the original noise spectral component. Hence, the variances (normalized power) of the two new noise components are smaller by a factor of 4. Accordingly, if the power spectral density of the original noise component at frequency $k\,\Delta f$ is $G_n(k\,\Delta f)$, then, from Eq. (7.37), the new components have spectral densities

$$G_n(k\,\Delta f + f_0) = G_n(k\Delta f - f_0) = \frac{G_n(k\,\Delta f)}{4} \tag{7.38}$$

In the limit as $\Delta f \to 0$, we replace $k\,\Delta f$ by the continuous variable f, and Eq. (7.38) becomes

$$G_n(f + f_0) = G_n(f - f_0) = \frac{G_n(f)}{4} \tag{7.39}$$

In words, given the power spectral density plot $G_n(f)$ of a noise waveform $n(t)$, the power spectral density of $n(t)\cos 2\pi f_0 t$ is arrived at as follows: divide $G_n(f)$ by 4, shift the divided plot to the left by amount f_0, to the right by amount f_0, and add the two shifted plots.

Now consider the following situation: We have noise $n(t)$ from which we single out two spectral components, one at frequency $k\,\Delta f$ and one at frequency $l\,\Delta f$. We mix with a sinusoid at frequency f_0, with f_0 selected to be midway between $k\,\Delta f$ and $l\,\Delta f$; that is, $f_0 = \frac{1}{2}(k + l)\,\Delta f$. Then the mixing will give rise to *four* spectral components, two difference-frequency components and two sum-frequency components. The two difference-frequency components will be at the same frequency $p\,\Delta f = f_0 - k\,\Delta f = l\,\Delta f - f_0$. However, we now show that these difference-frequency components are uncorrelated. Representing the spectral components at $k\,\Delta f$ and $l\,\Delta f$ as in Eqs (7.24) and (7.25), we find that the difference-frequency components are

$$n_{p1}(t) = \frac{a_k}{2} \cos 2\pi p\,\Delta f t - \frac{b_k}{2} \sin 2\pi p\,\Delta f t \tag{7.40}$$

and

$$n_{p2}(t) = \frac{a_l}{2} \cos 2\pi p\,\Delta f t + \frac{b_l}{2} \sin 2\pi p\,\Delta f t \tag{7.41}$$

where $n_{p1}(t)$ is the difference component due to the mixing of frequencies f_0 and $k\,\Delta f$, while $n_{p2}(t)$ is the difference component due to the mixing of frequencies f_0 and $l\,\Delta f$. If we now take into account, as we have established in Sec. 7.2.2, that $\overline{a_k a_l} = \overline{a_k b_l} = \overline{b_k a_l} = \overline{b_k b_l} = 0$, then we find from Eqs (7.40) and (7.41) that

$$E[n_{p1}(t) n_{p2}(t)] = 0 \tag{7.42}$$

we know that

Thus, as discussed in connection with Eq. (7.36), superposition of power applies, and the power at the difference frequency due to the superposition of $n_{p1}(t)$ and $n_{p2}(t)$ is

$$E\{[n_{p1}(t) + n_{p2}(t)]^2\} = E\{[n_{p1}(t)]^2\} + E\{[n_{p2}(t)]^2\} \tag{7.43}$$

Thus, mixing noise with a sinusoidal signal results in a frequency shifting of the original noise by f_0. The variance of the shifted noise is found by adding the variance of each new noise component. Thus we see that the principle stated immediately after Eq. (7.39) applies even when there is overlap in the two shifted power spectral density plots.