7.5 QUADRATURE COMPONENTS OF NOISE

We have represented noise $n(t)$, as in Eq. (7.7), as the superposition of spectral components of noise in the expression

$$n(t) = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} (a_k \cos 2\pi k \,\Delta f t + b_k \sin 2\pi k \,\Delta f t) \tag{7.78}$$

It is sometimes more advantageous to represent the noise through an alternative representation given by

$$n(t) = n_c(t) \cos 2\pi f_0 t - n_s(t) \sin 2\pi f_0 t \tag{7.79}$$

in which f_0 is an arbitrary frequency. The representation of Eq. (7.79) is frequently used with great convenience in dealing with noise confined to a relatively narrow frequency band in the neighborhood of f_0. For this reason Eq. (7.79) is often referred to as the *narrowband* representation. The term *quadrature component* representation is also often used because of the appearance in the equation of sinusoids in quadrature.

We may readily transform Eq. (7.78) into Eq. (7.79) and, in doing so, arrive at explicit expressions for $n_c(t)$ and $n_s(t)$. Let us select f_0 to correspond to $k = K$; that is, we set

$$f_0 = K \,\Delta f \tag{7.80}$$

Adding $2\pi f_0 t - 2\pi K \,\Delta f t = 0$ to the arguments in Eq. (7.78), we have

$$n(t) = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} \{a_k \cos 2\pi[f_0 + (k-K)\,\Delta f]t + b_k \sin 2\pi[f_0 + (k-K)\,\Delta f]t\} \tag{7.81}$$

Using the trigonometric identities for the cosine of the sum of two angles and for the sine of the sum of two angles, it is readily verified that $n(t)$ is indeed given by Eq. (7.79), provided that $n_c(t)$ and $n_s(t)$ are taken to be

$$n_c(t) = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} [a_k \cos 2\pi(k-K)\,\Delta f t + b_k \sin 2\pi(k-K)\,\Delta f t] \tag{7.82}$$

and

$$n_s(t) = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} [a_k \sin 2\pi(k-K)\,\Delta f t - b_k \cos 2\pi(k-K)\,\Delta f t] \tag{7.83}$$

Like $n(t)$, so also $n_c(t)$ and $n_s(t)$ are stationary random processes which are represented as linear superpositions of spectral components. We recall, from Sec. 7.2.2, that the a_k's and b_k's are Gaussian random variables of zero mean and equal variance and that, further, the a_k's and b_k's are uncorrelated. We may then readily establish (Probs. 7.79 to 7.81) that $n_c(t)$ and $n_s(t)$ are Gaussian random processes of zero mean value and of equal variance and that, further, $n_c(t)$ and $n_s(t)$ are uncorrelated.

To see the significance of the quadrature representation of noise, let us use it in connection with narrowband noise. We observe in Eqs (7.82) and (7.83) that a noise spectral component in $n(t)$ of frequency $f = k\,\Delta f$ gives rise in $n_c(t)$ and $n_s(t)$ to a spectral component of frequency $(k-K)\,\Delta f = f - f_0$

Suppose then that the noise $n(t)$ is narrowband, extending over a bandwidth B. And suppose that f_0 is selected midway in the frequency range of the noise. Then the spectrum of the noise $n(t)$ extends over the range $f_0 - B/2$ to $f_0 + B/2$. On the other hand, the spectrum of $n_c(t)$ and $n_s(t)$ extends over only the range from $-B/2$ to $B/2$. By way of example, if the noise $n(t)$ is confined to a frequency band of only 10 kHz centered around f_0 = 10 MHz, then while $n(t)$ is a superposition of spectral components around the 10 MHz frequency, $n_c(t)$ and $n_s(t)$ change only insignificantly during the time the sinusoid of frequency f_0 executes a full cycle.

In view of the slow variations of $n_c(t)$ and $n_s(t)$ relative to the sinusoid of frequency f_0, it is reasonable and useful to give the quadrature representation of noise an interpretation in terms of phasors and a phasor diagram. Thus, in Eq. (7.79) the term $n_c(t) \cos 2\pi f_0 t$ is of frequency f_0 and of relatively slowly varying amplitude $n_c(t)$. Similarly, the term $-n_s(t) \sin 2\pi f_0 t$ is in quadrature with the first term and has a relatively slowly varying amplitude $n_s(t)$. In a coordinate system rotating counterclockwise with angular velocity $2\pi f_0$, these phasors are as represented in Fig. 7.10. These two phasors of varying amplitude give rise to a resultant phasor of amplitude $r(t) = [n_c^2(t) + n_s^2(t)]^{1/2}$ which makes an angle

$$\theta(t) = \tan^{-1} [n_s(t)/n_c(t)] \tag{7.84}$$

with the horizontal. With the passage of time, the end point of this resultant phasor wanders about randomly over the phasor diagram.

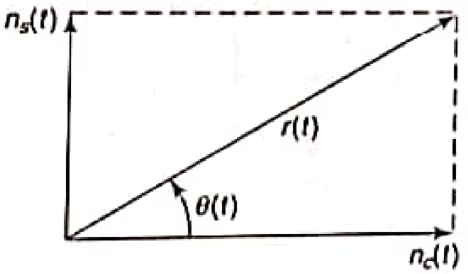

Fig. 7.10 *A phasor diagram of the quadrature representation of noise.*

We shall find the quadrature representation is useful generally in the analysis of noise and the phasor interpretation is especially useful in discussing angle modulation communications systems.