## 7.4 LINEAR FILTERING OF NOISE

Thermal noise has a power spectral density which is quite uniform up to frequencies of the order of $10^{13}$ Hz. Shot noise has a power spectral density which is reasonably constant up to frequencies which are of the order of the reciprocal of the transit time of charge carriers across the junction. Other noise sources similarly have very wide spectral ranges. We shall assume, in discussing the effect of noise on communication systems, that we have to contend with *white* noise. White noise is noise whose power spectral density is uniform over the entire frequency range of interest. The term *white* is used in analogy with white light, which is a superposition of all visible spectral components. Thus, we assume, as shown in Fig. 7.6, that over the entire spectrum, including positive and negative frequencies,

$$G_n(f) = \frac{\eta}{2} \tag{7.47}$$

in which $\eta$ is a constant (see Sec. 15.3).

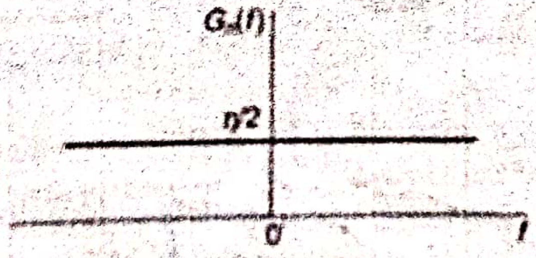


Fig. 7.6 Power spectral density of white noise.

In order to minimize the noise power that is presented to the demodulator of a receiving system, we introduce a filter before the demodulator as indicated in Fig. 7.7. The bandwidth $B$ of the filter is made as narrow as possible so as to avoid transmitting any unnecessary noise to the demodulator. For example, in an AM system in which the baseband extends to a frequency of $f_M$, the bandwidth $B = 2f_M$. In a wideband FM system the bandwidth is proportional to twice the frequency deviation. It is useful to consider the effect of certain types of filters on the noise. One of the filters most often used is the simple $RC$ low-pass filter.

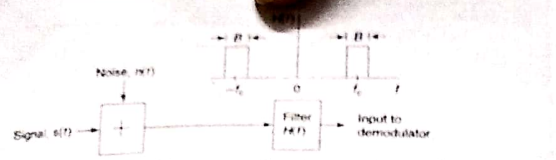


**Fig. 7.7** *A filter is placed before a demodulator to limit the noise power input to the demodulator.*

## The RC Low-pass Filter

An *RC* low-pass filter with a 3 dB frequency $f_c$ has the transfer function

$$H(f) = \frac{1}{1 + jf/f_c} \tag{7.48}$$

If the input noise to this filter has a power spectral density $G_{n_i}(f)$ and the power spectral density of the output noise is $G_{n_o}(f)$, then, using Eq. (7.34), we have

$$G_{n_o}(f) = G_{n_i}(f)|H(f)|^2 \tag{7.49}$$

If the noise is white, $G_{n_i}(f) = \eta/2$ for all frequencies, Eq. (7.49) becomes

$$G_{n_o}(f) = \frac{\eta}{2}\frac{1}{1+(f/f_c)^2} \tag{7.50}$$

The noise power at the filter output $N_o$ is

$$N_o = \int_{-\infty}^{\infty} G_{n_o}(f)\,df = \frac{\eta}{2}\int_{-\infty}^{\infty}\frac{df}{1+(f/f_c)^2} \tag{7.51}$$

Changing variables to $x = f/f_c$, and noting that $\int_{-\infty}^{\infty} dx/(1+x^2) = \pi$, we have

$$N_o = \frac{\pi}{2}\eta f_c \tag{7.52}$$

## The Rectangular (Ideal) Low-pass Filter

A *rectangular* low-pass filter has the transfer function

$$H(f) = \begin{cases} 1 & |f| \le B \\ 0 & \text{elsewhere} \end{cases} \tag{7.53}$$



Assuming that the noise input to the filter is *white*, the output-power spectral density is

$$G_{n_o}(f) = \begin{cases} \frac{\eta}{2} & -B \le f \le B \\ 0 & \text{elsewhere} \end{cases} \tag{7.54}$$

The output noise power is

$$N_o = \eta B \tag{7.55}$$

## A Rectangular Bandpass Filter

A rectangular bandpass filter is shown in Fig. 7.8. The bandwidth of the filter is $f_2 - f_1$. Then, with a *white* noise input, the output-noise power is

$$N_o = 2\frac{\eta}{2}(f_2 - f_1) = \eta(f_2 - f_1) \tag{7.56}$$

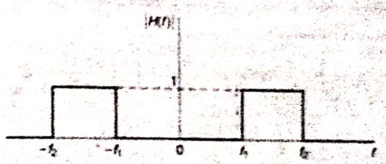


**Fig. 7.8** *A rectangular bandpass filter.*

## A Differentiating Filter

A differentiating filter is a network which yields at its output a waveform which is proportional to the time derivative of the input waveform. As discussed in Sec. 1.3.5, such a network has a transfer function $H(f)$ which is proportional to the frequency; that is,

$$H(f) = j2\pi\tau f \tag{7.57}$$

where $\tau$ is a constant factor of proportionality. If white noise with $G_{n_i}(f) = \eta/2$ is passed through such a filter, then the output-noise-power spectral density is

$$G_{n_o}(f) = |H(f)|^2 G_{n_i}(f) = 4\pi^2\tau^2 f^2 \frac{\eta}{2} \tag{7.58}$$

If the differentiator is followed by a rectangular low-pass filter having a bandwidth $B$, as described by Eq. (7.53), the noise power at the output of the low-pass filter is

$$N_o = \int_{-B}^{B} 4\pi^2\tau^2 f^2 \frac{\eta}{2}\,df = \frac{4\pi^2}{3}\eta\tau^2 B^3 \tag{7.59}$$

## An Integrator

Let noise $n(t)$ be applied to the input of an integrator at time $t = 0$. We calculate the noise power at the integrator output at a time $t = T$. The result will be of interest in connection with the discussion of the *matched filter* in Chap. 11.

A network which performs the operation of integration has a transfer function $1/j\omega\tau$. A delay by an interval $T$ is represented by a factor $e^{-j\omega T}$. Hence, a network which performs an integration over an interval $T$ may be represented by a network whose transfer function is

$$H(f) = \frac{1}{j\omega\tau} - \frac{e^{-j\omega T}}{j\omega\tau} = \frac{1 - e^{-j\omega T}}{j\omega\tau} \tag{7.60}$$

where $\tau$ is a constant. We find, with $\omega = 2\pi f$, that

$$|H(f)|^2 = \left(\frac{T}{\tau}\right)^2 \left(\frac{\sin \pi T f}{\pi T f}\right)^2 \tag{7.61}$$

The noise power output of such a filter with white input noise of power spectral density $\eta/2$ is (using $x \equiv \pi\tau f$)

$$N_o = \int_{-\infty}^{\infty} \frac{\eta}{2} |H(f)|^2 \, df = \frac{\eta}{2}\left(\frac{T}{\tau}\right)^2 \int_{-\infty}^{\infty} \left(\frac{\sin \pi T f}{\pi T f}\right)^2 df \tag{7.62a}$$

$$= \frac{\eta T}{2\pi\tau^2} \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx \tag{7.62b}$$

The definite integral in Eq. (7.62*b*) has the value $\pi$, so that finally

$$N_o = \frac{\eta T}{2\tau^2} \tag{7.63}$$

It is instructive to obtain Eq. (7.63) by a calculation in the time domain. If an input noise *sample* to the integrator is $n_i(t)$ and the corresponding output of the integrator is $n_o(T)$ then

$$n_o(T) = \frac{1}{\tau}\int_0^T n_i(t)\, dt \tag{7.64}$$

Note that $n_o(T)$ is a random variable since $T$ is a constant.

The expected value (ensemble average) of the random variable $n_o(T)$ is

$$m_{n_o} = E[n_o(T)] = E\left\{\frac{1}{\tau}\int_0^T n_i(t)\, dt\right\} \tag{7.65a}$$

In general, as discussed in Sec. 6.2.6, if $f(x)$ is the probability density of a random variable $x$, and $u(x)$ is some function of $x$, then

$$E[u(x)] = \int_{-\infty}^{\infty} u(x) f(x)\, dx \tag{7.65b}$$



Correspondingly, we have in the present case, that

$$m_{n_o} = \int_{-\infty}^{\infty} dn_i\, f(n_i) \left\{\frac{1}{\tau}\int_0^T n_i(t)\, dt\right\} \tag{7.65c}$$

Interchanging the order of integration (see Prob. 7.15) yields

$$m_{n_o} = \frac{1}{\tau}\int_0^T dt \left\{\int_{-\infty}^{\infty} n_i\, f(n_i)\, dn_i\right\} \tag{7.66}$$

But the integral in brackets is the mean value of $n_i$, i.e.

$$m_{n_i} = E(n_i) = \int_{-\infty}^{\infty} n_i f(n_i) dn_i \tag{7.67}$$

Assuming, as before, that the average value of the input noise $m_{n_i} = 0$, we see that the average value of the output noise $m_{n_o} = 0$ as well.

The normalized noise power $N_o$ corresponding to the random variable $n_o(T)$ is equal to the variance of $n_o(T)$ (see Sec. 6.2.7). Thus,

$$N_o = \sigma_{n_o}^2 = E\{[n_o(T)]^2\} = E\left\{\frac{1}{\tau^2}\int_0^T n_i(t)\, dt \cdot \int_0^T n_i(\lambda)\, d\lambda\right\} \tag{7.68}$$

where $t$ and $\lambda$ are dummy variables of integration. $N_o$ can be rewritten as

$$N_o = E\left\{\frac{1}{\tau^2}\int_0^T \int_0^T n_i(t) n_i(\lambda) dt\, d\lambda\right\} \tag{7.69}$$

If we define

$$\alpha \equiv n_i(t) \qquad \text{and} \qquad \beta \equiv n_i(\lambda)$$

Equation (7.69) can be written as

$$N_o = \frac{1}{\tau^2}\int_{-\infty}^{\infty} f(\alpha, \beta)\left\{\int_0^T dt \int_0^T d\lambda\, \alpha\beta\right\} d\alpha\, d\beta \tag{7.70}$$

Interchanging the order of integration (see Prob. 7.16) yields

$$N_o = \frac{1}{\tau^2}\int_0^T dt \int_0^T d\lambda \int_{-\infty}^{\infty} \alpha\beta f(\alpha, \beta)\, d\alpha\, d\beta \tag{7.71}$$

But

$$R_{n_i}(t - \lambda) = E[n_i(t) n_i(\lambda)] = E(\alpha\beta) = N_o \tag{7.72}$$

where $R_{n_i}(t - \lambda)$ is the autocorrelation function of $n_i(t)$.

We recall (see Eq. 6.141) that the correlation and the power spectral density are Fourier transform pairs. Since for white noise the power spectral density is $G(f) = \eta/2$, we have

$$E[n_i(t) n_i(\lambda)] = R_{n_i}(t - \lambda) = \mathcal{F}\{G(f)\} = \int_{-\infty}^{\infty} \eta/2\, e^{j2\pi f(t-\lambda)}\, df = (\eta/2)\, \delta(t - \lambda) \tag{7.73}$$