Suppose then that the noise $n(t)$ is narrowband, extending over a bandwidth B. And suppose that f_0 is selected midway in the frequency range of the noise. Then the spectrum of the noise $n(t)$ extends over the range $f_0 - B/2$ to $f_0 + B/2$. On the other hand, the spectrum of $n_c(t)$ and $n_s(t)$ extends over only the range from $-B/2$ to $B/2$. By way of example, if the noise $n(t)$ is confined to a frequency band of only 10 kHz centered around f_0 = 10 MHz, then while $n(t)$ is a superposition of spectral components around the 10 MHz frequency, $n_c(t)$ and $n_s(t)$ change only insignificantly during the time the sinusoid of frequency f_0 executes a full cycle.

In view of the slow variations of $n_c(t)$ and $n_s(t)$ relative to the sinusoid of frequency f_0, it is reasonable and useful to give the quadrature representation of noise an interpretation in terms of phasors and a phasor diagram. Thus, in Eq. (7.79) the term $n_c(t) \cos 2\pi f_0 t$ is of frequency f_0 and of relatively slowly varying amplitude $n_c(t)$. Similarly, the term $-n_s(t) \sin 2\pi f_0 t$ is in quadrature with the first term and has a relatively slowly varying amplitude $n_s(t)$. In a coordinate system rotating counterclockwise with angular velocity $2\pi f_0$, these phasors are as represented in Fig. 7.10. These two phasors of varying amplitude give rise to a resultant phasor of amplitude $r(t) = [n_c^2(t) + n_s^2(t)]^{1/2}$ which makes an angle

$$\theta(t) = \tan^{-1} [n_s(t)/n_c(t)] \tag{7.84}$$

with the horizontal. With the passage of time, the end point of this resultant phasor wanders about randomly over the phasor diagram.

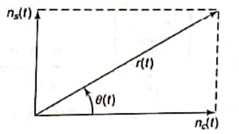

Fig. 7.10 A phasor diagram of the quadrature representation of noise.

We shall find the quadrature representation is useful generally in the analysis of noise and the phasor interpretation is especially useful in discussing angle modulation communications systems.

7.5.1 Power Spectral Density of Quadrature Components

To determine the power spectral density of $n_c(t)$, let us select from the original noise $n(t)$ those spectral components corresponding to $k = K + \lambda$ and $k = K - \lambda$, where λ, like k and K, is an integer. Since $k = K$ corresponds to the frequency f_0, the selected components correspond to frequencies $f_0 + \lambda\,\Delta f$ and $f_0 - \lambda\,\Delta f$. These two frequencies give rise to four power spectral lines in a two-sided power spectral density plot as shown in Fig. 7.11. In this figure, we have assumed band-limited noise. However, for the sake of generality we have not assumed that the power spectral density is uniform in the band, nor have we assumed that the frequency $f_0 = K\,\Delta f$ is located at the center of the band.

We now select from $n_c(t)$, as given by Eq. (7.82), that part, $\Delta n_c(t)$, corresponding to our selection of frequencies, $f_0 \pm \lambda\,\Delta f$, from $n(t)$. We find

$$\Delta n_c(t) = a_{K-\lambda} \cos 2\pi\lambda\,\Delta f t - b_{K-\lambda} \sin 2\pi\lambda\,\Delta f t + a_{k+\lambda} \cos 2\pi\lambda + \Delta f t + b_{K+\lambda} \sin 2\pi\lambda\,\Delta f t \tag{7.85}$$

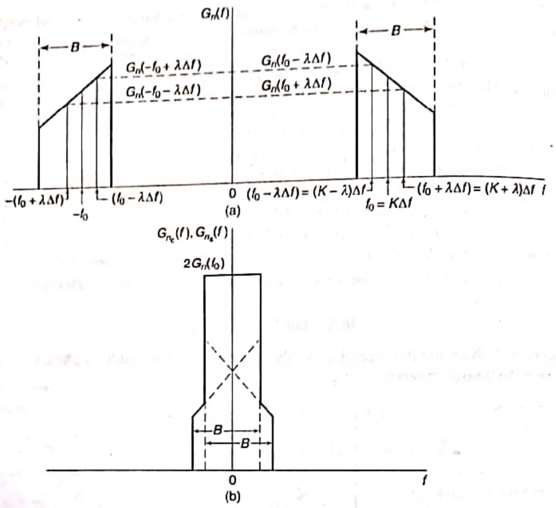

Fig. 7.11 (a) Power spectrum of bandlimited noise. (b) Power spectral density of n_c and n_s.

Note that all four terms in Eq. (7.85) are of the same frequency $\lambda\,\Delta f$. These four terms represent four uncorrelated random processes, since the a's and b's are uncorrelated random variables. Hence we may find the power P_λ of $\Delta n_c(t)$ by determining the ensemble average of $[\Delta n_c(t)]^2$. Since $\Delta n_c(t)$ is a stationary random process, the ensemble average can be calculated at any time $t = t_1$. Following the procedure employed in Sec. 7.2.2, we choose t_1 so that $\lambda\,\Delta f t_1$ is an integer. Then

$$\Delta n_c(t_1) = a_{K-\lambda} + a_{K+\lambda} \tag{7.86}$$

and

$$P_\lambda = E\{[\Delta n_c(t_1)]^2\} = E[(a_{K-\lambda} + a_{K+\lambda})^2] \tag{7.87}$$

Using Eq. (7.26), which says that $E(a_{K-\lambda} a_{K+\lambda}) = 0$, we have, from Eq. (7.87), that

$$P_\lambda = \overline{a_{K-\lambda}^2} + \overline{a_{K+\lambda}^2} \tag{7.88}$$

From Eqs (7.20) and (7.88), we then find

$$P_\lambda = 2G_{n_c}(\lambda\,\Delta f)\,\Delta f = 2G_n[(K-\lambda)\,\Delta f]\,\Delta f + 2G_n[(K+\lambda)\,\Delta f]\,\Delta f \tag{7.89}$$

Hence,

$$G_{n_c}(\lambda\,\Delta f) = G_n[(K-\lambda)\,\Delta f] + G_n[(K+\lambda)\,\Delta f] \tag{7.90}$$

We now set $K\,\Delta f = f_0$ and replace $\lambda\,\Delta f$ by a continuous frequency variable f, and we have, from Eq. (7.90),

$$G_{n_c}(f) = G_n(f_0 - f) + G_n(f_0 + f) \tag{7.91}$$

In a similar manner we may deduce an identical result for $G_{n_s}(f)$, namely,

$$G_{n_s}(f) = G_n(f_0 - f) + G_n(f_0 + f) \tag{7.92}$$

Expressed in words, Eqs (7.91) and (7.92) say, that to find the power spectral density of $n_c(t)$ or of $n_s(t)$ at a frequency f, add the power spectral densities of $n(t)$ at the frequencies $f_0 - f$ and $f_0 + f$. In view of this result, and in view of the symmetry of a two-sided power spectral density plot as in Fig. 7.11, it may readily be verified that the plot of $G_{n_c}(f)$ or $G_{n_s}(f)$ may be constructed from the plot of $G_n(f)$ in the following manner:

1. Displace the positive-frequency portion of the plot of $G_n(f)$ to the left by amount f_0 so that the portion of the plot originally located at f_0 is now coincident with the ordinate.
2. Displace the negative-frequency portion of the plot of $G_n(f)$ to the right by an amount f_0.
3. Add the two displaced plots. The result of applying this procedure to the plot of Fig. 7.11*a* is shown in Fig. 7.11*b*.

A case of special interest is considered in the following example.