We now set $K\,\Delta f = f_0$ and replace $\lambda\,\Delta f$ by a continuous frequency variable f, and we have, from Eq. (7.90),

$$G_{n_c}(f) = G_n(f_0 - f) + G_n(f_0 + f) \tag{7.91}$$

In a similar manner we may deduce an identical result for $G_{n_s}(f)$, namely,

$$G_{n_s}(f) = G_n(f_0 - f) + G_n(f_0 + f) \tag{7.92}$$

Expressed in words, Eqs (7.91) and (7.92) say, that to find the power spectral density of $n_c(t)$ or of $n_s(t)$ at a frequency f, add the power spectral densities of $n(t)$ at the frequencies $f_0 - f$ and $f_0 + f$. In view of this result, and in view of the symmetry of a two-sided power spectral density plot as in Fig. 7.11, it may readily be verified that the plot of $G_{n_c}(f)$ or $G_{n_s}(f)$ may be constructed from the plot of $G_n(f)$ in the following manner:

1. Displace the positive-frequency portion of the plot of $G_n(f)$ to the left by amount f_0 so that the portion of the plot originally located at f_0 is now coincident with the ordinate.
2. Displace the negative-frequency portion of the plot of $G_n(f)$ to the right by an amount f_0.
3. Add the two displaced plots. The result of applying this procedure to the plot of Fig. 7.11*a* is shown in Fig. 7.11*b*.

A case of special interest is considered in the following example.

Example 7.5

White noise with power spectral density $\eta/2$ is filtered by a rectangular bandpass filter with $H(f) = 1$, centered at f_0 and having a bandwidth B. Find the power spectral density of $n_c(t)$ and $n_s(t)$. Calculate the power in $n_c(t)$, $n_s(t)$ and $n(t)$.

Solution

Since the filter is rectangular with $H(f) = 1$, the power spectral density of the output noise $n(t)$ is

$$G_n(f) = \begin{cases} \dfrac{\eta}{2} & f_0 - \dfrac{B}{2} \le |f| \le f_0 + \dfrac{B}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{7.93}$$

Hence, $G_n(f_0 + f) = G_n(f_0 - f)$, and the spectral density of $n_c(t)$ and $n_s(t)$ is

$$G_{n_c}(f) = G_{n_s}(f) = G_n(f_0 - f) + G_n(f_0 + f) = \frac{\eta}{2} + \frac{\eta}{2} = \eta \qquad |f| \le \frac{B}{2} \tag{7.94}$$

Note the extremely important result that the magnitude of $G_{n_c}(f) = G_{n_s}(f)$ is *twice* the magnitude of $G_n(f_0 + f)$.

The power (variance) of $n_c(t)$, and of $n_s(t)$, is

$$\sigma_{n_c}^2 = \sigma_{n_s}^2 = \int_{-B/2}^{B/2} G_{n_c}(f)\,df = \eta B \tag{7.95}$$

The power (variance) of $n(t)$ is

$$\sigma_n^2 = \int_{-f_0 - B/2}^{-f_0 + B/2} G_n(f)\,df + \int_{f_0 - B/2}^{f_0 + B/2} G_n(f)\,df = 2\frac{\eta}{2}B = \eta B \tag{7.96}$$

Thus, the power of $n_c(t)$, $n_s(t)$ and $n(t)$ are each equal.

7.5.2 Probability Density of Quadrature Components and Time Derivatives

We have noted that $n_c(t)$ and $n_s(t)$ are Gaussian random processes with mean values of zero. If the noise $n(t)$ has a power spectral density $\eta/2$ over a bandwidth B, then, as noted in the preceding

example, $\sigma_{n_c}^2 = \sigma_{n_s}^2 = \eta B$. Using Eq. (7.85) with $m = 0$, we find that the probability densities of the random variables n_c and n_s [that is, $n_c(t)$ and $n_s(t)$ at any fixed time] are given by

$$f(n_c) = \frac{1}{\sqrt{2\pi\eta B}} e^{-n_c^2/2\eta B} \tag{7.97a}$$

$$f(n_s) = \frac{1}{\sqrt{2\pi\eta B}} e^{-n_s^2/2\eta B} \tag{7.97b}$$

Since $n_c(t)$ and $n_s(t)$ are Gaussian, the time derivatives $\dot{n}_c(t)$ and $\dot{n}_s(t)$ are also Gaussian, because the operation of differentiation is an operation performed by a linear filter (Eq. 7.57), and from Sec. 7.2.1 we know that filtering Gaussian noise does not change its probability density. To write the probability densities of $\dot{n}_c(t)$ and $\dot{n}_s(t)$, we first evaluate their variances $\sigma_{\dot{n}_c}^2$ and $\sigma_{\dot{n}_s}^2$. Noting that differentiation is equivalent to multiplying each spectral component by $j\omega$, we find

$$G_{\dot{n}_c}(f) = |j\omega|^2\, G_{n_c}(f) = 4\pi^2 f^2 G_{n_c}(f) \tag{7.98}$$

so that, using Eq. (7.94), we find

$$\sigma_{\dot{n}_c}^2 = \int_{-B/2}^{B/2} G_{\dot{n}_c}(f)\,df = \int_{-B/2}^{B/2} 4\pi^2 f^2 \eta\,df = \frac{\pi^2}{3}\eta B^3 \tag{7.99}$$

with an identical result for $\sigma_{\dot{n}_s}^2$. Hence, we find

$$f(\dot{n}_c) = \frac{\exp\{-\dot{n}_c^2/[(2\pi^2/3)\eta B^3]\}}{\sqrt{(2\pi^3/3)\eta B^3}} \tag{7.100}$$

with a similar expression for $f(\dot{n}_s)$. Assuming that the four random variables n_c, n_s, $\dot{n}_c$ and $\dot{n}_s$ are independent, the joint distribution function for the four variables is the product of the individual densities. Hence, from Eqs (7.97) and (7.99) we find

$$f(n_c, n_s, \dot{n}_c, \dot{n}_s) = \frac{\exp[-(n_c^2 + n_s^2)/2\eta B]\exp[-(\dot{n}_c^2 + \dot{n}_s^2)/(2\pi^2/3)\eta B^3]}{[(2\pi^2/\sqrt{3})\eta B^2]^2} \tag{7.101}$$

We shall have occasion to use Eq. (7.101) in Chap. 10 in connection with an analysis of threshold effects in frequency modulation.

In arriving at Eq. (7.101), we assumed that the four random variables involved were independent. That such is indeed the case may be verified in the manner indicated in Prob. 7.28.

7.6 REPRESENTATION OF NOISE USING ORTHONORMAL COORDINATES

In our discussion of the frequency-domain representation of noise we saw that a noise process can be represented as a sum of orthonormal functions. These orthonormal functions are the sines and cosines. In our discussion of the Gram–Schmitt technique, where our interest concerned a waveform